ON THE ABELIANIZATION OF CONGRUENCE SUBGROUPS OF $\mathrm{Aut}(F_2)$

DANIEL APPEL

ABSTRACT. Let F_n be the free group of rank n and let $\mathrm{Aut}^+(F_n)$ be its special automorphism group. For an epimorphism $\pi : F_n \to G$ of the free group F_n onto a finite group G we call $\Gamma^+(G, \pi) = \{\varphi \in \mathrm{Aut}^+(F_n) \mid \pi\varphi = \pi\}$ the standard congruence subgroup of $\mathrm{Aut}^+(F_n)$ associated to G and π. In the case $n = 2$ we fully describe the abelianization of $\Gamma^+(G, \pi)$ for finite abelian groups G. Moreover, we show that if G is a finite non-perfect group, then $\Gamma^+(G, \pi) \leq \mathrm{Aut}^+(F_2)$ has infinite abelianization.

1. Introduction

1.1. **Main Results.** Let F_n be the free group on n generators and $\mathrm{Aut}(F_n)$ its group of automorphisms. Moreover, let $\pi : F_n \to G$ be an epimorphism of F_n onto a finite group G. The automorphism group $\mathrm{Aut}(F_n)$ acts in a natural way on the (finite) set of all epimorphisms from F_n onto G. By

$$\Gamma(G, \pi) := \{\varphi \in \mathrm{Aut}(F_n) \mid \pi\varphi = \pi\}$$

we denote the stabilizer of π under this action. The group $\Gamma(G, \pi)$ is called the *standard congruence subgroup of* $\mathrm{Aut}(F_n)$ *associated to* G *and* π. This is a finite index subgroup of $\mathrm{Aut}(F_n)$. A subgroup of $\mathrm{Aut}(F_n)$ containing some $\Gamma(G, \pi)$ is called a *congruence subgroup of* $\mathrm{Aut}(F_n)$.

A classical question is whether every finite index subgroup of $\mathrm{Aut}(F_n)$ is a congruence subgroup. Quite recently it has been shown that every finite index subgroup of $\mathrm{Aut}(F_2)$ is a congruence subgroup. See [2] and [3]. For $n \geq 3$ this question is still open.

Date: February 12, 2010.

The author would like to thank F. Grunewald for proposing the topic and B. Klopsch for many helpful discussions.

The groups $\Gamma(G, \pi)$ have been studied by various authors. For instance, in [5], F. Grunewald and A. Lubotzky use the groups $\Gamma(G, \pi)$ to construct linear representations of the automorphism group $\mathrm{Aut}(F_n)$. Our work is related to results of T. Satoh [11, 12], see also Section 1.2. The joint work [1] of E. Ribnere and the author can be seen as an accompanying paper to the present one.

The automorphism group $\mathrm{Aut}(F_n)$ has a well-known representation onto $\mathrm{GL}_n(\mathbb{Z})$ given by

$$\rho : \mathrm{Aut}(F_n) \longrightarrow \mathrm{Aut}(F_n/F_n') \cong \mathrm{GL}_n(\mathbb{Z}),$$

where F_n' denotes the commutator subgroup of F_n, see [8, Sec. 3.6] for details. Its kernel is denoted by IA_n and called the *group of* IA_n-*automorphisms* or sometimes also the *classical Torelli group*. For an interesting generalization see [13].

As one classically considers $\mathrm{SL}_n(\mathbb{Z})$ instead of $\mathrm{GL}_n(\mathbb{Z})$, we focus on the *special automorphism group* $\mathrm{Aut}^+(F_n) := \rho^{-1}(\mathrm{SL}_n(\mathbb{Z}))$, which is a subgroup of index 2 in $\mathrm{Aut}(F_n)$. We set $\Gamma^+(G, \pi) := \Gamma(G, \pi) \cap \mathrm{Aut}^+(F_n)$. This is a subgroup of index at most 2 in $\Gamma(G, \pi)$. The term *congruence subgroup of* $\mathrm{Aut}^+(F_n)$ is defined in the obvious way.

In this paper we study the abelianizations $\Gamma^+(G, \pi)^{\mathrm{ab}}$ of the groups $\Gamma^+(G, \pi)$ in the case $n = 2$. We remark that if G is abelian, then, up to conjugation, $\Gamma^+(G, \pi)$ depends only on G but not on the particular epimorphism $\pi : F_2 \to G$, see [1, Lem. 3.1]. Moreover, every finite abelian group generated by two elements can be written as $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ where $n, m \in \mathbb{N}$ such that $n \mid m$. The following theorem therefore covers all possible choices for an epimorphism $\pi : F_2 \to G$ onto a finite abelian group.

Theorem 1.1. *Let $m, n \in \mathbb{N}$ such that $m \geq 3$, $n \mid m$ and $(m, n) \neq (3, 1)$. Let $G := \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and $\pi : F_2 \to G$ be an epimorphism. Then*

$$\Gamma^+(G, \pi)^{\mathrm{ab}} \cong G \times \mathbb{Z}^{1+12^{-1}nm^2 \prod_{p \mid m}(1-p^{-2})}$$

where the product runs over all primes p dividing m.

Furthermore, we have

$$\Gamma^+(\mathbb{Z}/2\mathbb{Z}, \pi)^{\mathrm{ab}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z},$$

$$\Gamma^+(\mathbb{Z}/3\mathbb{Z}, \pi)^{\mathrm{ab}} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z},$$

$$\Gamma^+(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \pi)^{\mathrm{ab}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^2.$$

Theorem 1.2. *Let $\pi : F_2 \to G$ be an epimorphism of F_2 onto a finite non-perfect group G. Then $\Gamma^+(G,\pi)$ has infinite abelianization.*

For $n, m \in \mathbb{N}$ with $n \mid m$ we define a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ by

$$\Gamma(m,n) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv_m 1, b \equiv_m 0 \text{ and } c \equiv_n 0, d \equiv_n 1 \}.$$

By $\mathrm{P}\Gamma(m,n)$ we denote the image of $\Gamma(m,n)$ in $\mathrm{PSL}_2(\mathbb{Z})$ under the natural projection. One of the main ingredients in our proofs is

Proposition 1.3. *Let $m, n \in \mathbb{N}$ such that $m \geq 3$, $n \mid m$ and $(m,n) \neq (3,1)$. Then $\Gamma(m,n)$ and $\mathrm{P}\Gamma(m,n)$ are free of rank*

$$1 + \frac{nm^2}{12} \prod_{\substack{p \mid m \\ p \text{ prime}}} \left(1 - \frac{1}{p^2}\right).$$

In particular, for primes $p \geq 5$, the groups $\Gamma(p,1) \leq \mathrm{SL}_2(\mathbb{Z})$ are free of rank $1 + \frac{1}{12}p^2(1 - p^{-2})$ so that the rank of $\Gamma(p,1)$ grows quadratically in p. In contrast, for $n \geq 3$, one can show that the corresponding subgroups in $\mathrm{SL}_n(\mathbb{Z})$ can always be generated by $n(n-1)$ matrices. See [5, Lem. 6.1].

1.2. **Related Results and Open Problems.** In [5] Grunewald and Lubotzky use the groups $\Gamma(G,\pi)$ to construct linear representations of the automorphism group $\mathrm{Aut}(F_n)$. In their concluding Section 9.4 they present, for some explicit G of small order, the indices of the groups $\Gamma^+(G,\pi)$ in $\mathrm{Aut}^+(F_n)$ and also the abelianizations of the groups $\Gamma^+(G,\pi)$ which they obtain by MAGMA computations. Besides the case that G is finite abelian, they also consider the case that $G = D_r$ is a dihedral group. Their observations are explained by another result of the author, which says

$$\Gamma^+(D_r,\pi)^{\mathrm{ab}} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^2, & r \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^3, & r \text{ even} \end{cases}.$$

The proof of this result is elementary but rather long. It shall therefore be postponed to the author's Ph.D. Thesis.

Actually, Grunewald and Lubotzky present computational results for some more finite groups G, e.g., $G = A_5$. In all considered cases $\Gamma^+(G,\pi)$ has infinite abelianization. For G non-perfect we now know by Theorem 1.2 that $\Gamma^+(G,\pi)^{\mathrm{ab}}$ is infinite. However, our proof does not work for perfect groups G. Hence we state

Conjecture 1.4. *For every epimorphism $\pi : F_2 \to G$ onto a non-trivial finite group G the group $\Gamma^+(G,\pi) \leq \mathrm{Aut}^+(F_2)$ has infinite abelianization.*

The situation in the case $n \geq 3$ looks different. Indeed, Grunewald and Lubotzky show that for every epimorphism $\pi : F_n \to G$ from F_n, $n \geq 3$, onto a finite abelian group, the group $\Gamma(G,\pi) \leq \mathrm{Aut}(F_n)$ has finite abelianization. This is Proposition 8.5 in [5]. Computational results [5, Sec. 9.4] indicate that $\Gamma^+(G,\pi)$ always has finite abelianization if $n \geq 3$. This leads to

Problem 1.5. *Does $\Gamma^+(G,\pi) \leq \mathrm{Aut}^+(F_n)$, where $n \geq 3$, have finite abelianization for every epimorphism $\pi : F_n \to G$ onto a non-trivial finite group G?*

Our work is related to results of Satoh [11, 12]. In his papers Satoh considers the kernel $T_{n,m}$ of the composition

$$\mathrm{Aut}(F_n) \xrightarrow{\rho} \mathrm{GL}_n(\mathbb{Z}) \longrightarrow \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z}).$$

One easily sees that for $m \geq 3$ we have $T_{n,m} = \Gamma^+((\mathbb{Z}/m\mathbb{Z})^n, \pi)$ where $\pi : F_n \to (\mathbb{Z}/m\mathbb{Z})^n$ is the obvious epimorphism. Satoh shows that for $n, m \geq 2$ one has

$$T_{n,m}^{\mathrm{ab}} \cong (\mathrm{IA}_n^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}) \times \Gamma_n(m)^{\mathrm{ab}}$$

where $\Gamma_n(m)$ is the kernel of the natural epimorphism $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$. Since IA_2 is free of rank 2, see [8, Sec. 3.6, Cor. N4], for $n = 2$ this reads

$$T_{2,m}^{\mathrm{ab}} \cong (\mathbb{Z}/m\mathbb{Z})^2 \times \Gamma_2(m)^{\mathrm{ab}}.$$

Observe that for $m \geq 3$ we have $\Gamma_2(m) = \Gamma(m,m) \leq \mathrm{SL}_2(\mathbb{Z})$. This result therefore corresponds to our result in Theorem 1.1 for the special case $G = (\mathbb{Z}/m\mathbb{Z})^2$. Satoh also gives the integral homology groups of $T_{2,p}$ for odd primes p. In particular, he shows that

$$\mathrm{H}_1(T_{2,p}, \mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^2 \times \mathbb{Z}^{1+12^{-1}p^3(1-p^{-2})}.$$

Since the first integral homology group is just the abelianization, this corresponds to our result in Theorem 1.1 for the special case $G = (\mathbb{Z}/p\mathbb{Z})^2$.

2. Congruence Subgroups of $\mathrm{SL}_2(\mathbb{Z})$

2.1. **Introduction.** Let $\pi : F_2 \to G$ be an epimorphism of the free group F_2 onto a finite group G. In order to understand the image $\rho(\Gamma^+(G,\pi))$ in $\mathrm{SL}_2(\mathbb{Z})$, we introduce some families of finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Recall that for two natural numbers $m, n \in \mathbb{N}$ such that $n \mid m$ we define the group

$$\Gamma(m,n) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv_m 1, b \equiv_m 0 \text{ and } c \equiv_n 0, d \equiv_n 1\}.$$

For $m \in \mathbb{N}$ the group $\Gamma(m) := \Gamma(m,m)$ is called the *principal congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *of level* m. It is the kernel of the natural epimorphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$. A subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing some $\Gamma(m)$ is called a *congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$. It is a classical result that not all finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are congruence subgroups. See for example [7].

In [1] it is shown that

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(m,n)] = nm^2 \prod_{p \mid m} \left(1 - \frac{1}{p^2}\right) \tag{2.1}$$

where the product runs over all primes p dividing m.

It is convenient to use the following notation.

$$\begin{aligned} \Gamma^0(m) &:= \{A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right) \mod m\}, \\ \Gamma^1(m) &:= \{A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv \left(\begin{smallmatrix} 1 & 0 \\ * & * \end{smallmatrix}\right) \mod m\}. \end{aligned}$$

All the above subgroups of $\mathrm{SL}_n(\mathbb{Z})$ are clearly congruence subgroups.

We denote the images in $\mathrm{PSL}_2(\mathbb{Z})$ of $\Gamma^0(m)$, $\Gamma^1(m)$ and $\Gamma(m)$ under the natural projection $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z})$ by $\mathrm{P}\Gamma^0(m)$, $\mathrm{P}\Gamma^1(m)$ and $\mathrm{P}\Gamma(m)$, respectively.

2.2. **Free Congruence Subgroups.** In [4] H. Frasch gives the following description of the groups $\mathrm{P}\Gamma(p)$ for p prime.

Theorem 2.1 (Frasch)**.** *Let $p \geq 3$ be a prime. Then $\mathrm{P}\Gamma(p)$ is free of rank $1 + \frac{1}{12}p^3(1-p^{-2})$. Moreover $\mathrm{P}\Gamma(2)$ is free of rank 2.*

We shall now generalize his result. Consider the natural projection $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z})$. By definition it maps $\Gamma(m)$ onto $\mathrm{P}\Gamma(m)$. For $m \geq 3$ the kernel $\langle\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\rangle$ of this projection has trivial intersection

with $\Gamma(m)$. Hence we obtain an isomorphism

$$\Gamma(m) \xrightarrow{\cong} \mathrm{P}\Gamma(m).$$

Note that this argument does not work for the case $m = 2$. Indeed, in contrast to $\mathrm{P}\Gamma(2)$, the group $\Gamma(2)$ is not free but the direct product of a rank 2 free group and a cyclic group of order 2. This is the reason for the third exceptional case in Theorem 1.1

Lemma 2.2. *Let $m \geq 3$. Then $\Gamma(m) \cong \mathrm{P}\Gamma(m)$ is free of rank*

$$1 + \frac{m^3}{12} \prod_{p|m} \left(1 - \frac{1}{p^2}\right)$$

where the product runs over all primes p dividing m.

Proof. Observe that for $m_1, m_2 \in \mathbb{N}$ such that $m_1 \mid m_2$ we have $\Gamma(m_2) \leq \Gamma(m_1)$. Accordingly, the main point in our proof is that a subgroup of a free group is again free and its rank is given by the Schreier Formula [8, Thm. 2.10]. Since $\Gamma(2)$ is not free, we consider two cases.

Case 1: We have $m = 2^a$ for some $a \geq 2$. One can verify that $\mathrm{P}\Gamma(4)$ has index 4 in $\mathrm{P}\Gamma(2)$. Since the latter group is free of rank 2, we find that $\mathrm{P}\Gamma(4)$ is free of rank 5 and hence so is $\Gamma(4)$. From (2.1) we know that $\Gamma(2^a)$ has index 2^{3a-6} in $\Gamma(4)$. We thus find that $\Gamma(2^a)$ is free of rank $1 + \frac{1}{12} 2^{3a}(1 - 2^{-2})$, as claimed.

Case 2: We have $p_0 \mid m$ for some prime $p_0 > 2$. Again by (2.1), we see that

$$[\Gamma(p_0) : \Gamma(m)] = \frac{m^3}{p_0^3} \prod_{\substack{p|m \\ p \neq p_0}} \left(1 - \frac{1}{p^2}\right).$$

Since, by Proposition 2.1, $\Gamma(p_0)$ is free of rank $1 + \frac{1}{12} p_0^3 (1 - p_0^{-2})$, we obtain the desired result. $\square$

We next wish to generalize this result even further to the groups $\Gamma(m, n)$ where $m \geq 3$, $n \mid m$ and $(m, n) \neq (3, 1)$. The first step is, of course, to show that these groups are free. Here we use the well-known description of $\mathrm{PSL}_2(\mathbb{Z})$ as a free product

$$\mathrm{PSL}_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle * \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle$$

where the first factor has order 2 and the second one has order 3. From the Kurosh Subgroup Theorem [8, Cor. 4.9.1] it follows that every non-trivial element of finite order in $\mathrm{PSL}_2(\mathbb{Z})$ has either order 2 or 3.

Using this observation and considering some minimal cases, we obtain

Lemma 2.3. *Let $m \geq 4$, then $\mathrm{P}\Gamma^1(m)$ is a free group.*

Proof. First we consider the case that m has a prime factor $p \geq 5$. Note that it suffices to show that $\mathrm{P}\Gamma^1(p)$ is free. To this end, we show that $\mathrm{P}\Gamma^1(p)$ does not contain a non-trivial element of finite order. Then the Kurosh Subgroup Theorem yields the desired result. Assume that $A \in \mathrm{P}\Gamma^1(p)$ is a non-trivial element of finite order. By definition of $\mathrm{P}\Gamma^1(p)$ we have

$$A \equiv \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \mod p$$

for some $0 \leq k \leq p-1$. It follows that

$$A^p \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod p.$$

Now we consider two cases.

Case 1: $A^p = 1$. Then the order of A divides p and we have a contradiction, since A has either order 2 or 3.

Case 2: $A^p \neq 1$. Then A^p is a non-trivial element of $\mathrm{P}\Gamma(p)$, which is, by Theorem 2.1, a free group. Hence A^p does not have finite order, contradiction.

To prove the result for m not having a prime factor ≥ 5, it suffices to consider the cases where $m = 4, 6, 9$. By an explicit computation, using the Reidemeister-Schreier Method, one verifies that $\mathrm{P}\Gamma^1(m)$ is also free in these cases. $\square$

From Lemmas 2.2, 2.3, the formulas (2.1) and the Schreier Formula, we obtain Proposition 1.3.

H. Rademacher [10] gives a very explicit description of the groups $\mathrm{P}\Gamma^0(p)$ for p prime. Observe that for $p \in \{2, 3\}$ the groups $\mathrm{P}\Gamma^0(p)$ and $\mathrm{P}\Gamma^1(p)$ coincide. By two examples of Rademacher we have

$$\mathrm{P}\Gamma^1(2) \cong \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$$
$$\mathrm{P}\Gamma^1(3) \cong \mathbb{Z} * \mathbb{Z}/3\mathbb{Z}.$$

In particular, $\mathrm{P}\Gamma^1(2)$ and $\mathrm{P}\Gamma^1(3)$ are not free. This is the reason for the first two exceptional cases in Theorem 1.1.

3. Proofs of the Main Results

3.1. **Proof of Theorem 1.1.** Let $G = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ where $m \geq 3$, $n \mid m$ and $(m,n) \neq (3,1)$. Then, by [1, Lem. 3.1], up to conjugation, the group $\Gamma^+(G,\pi)$ only depends on G but not on the particular choice of the epimorphism $\pi : F_2 \to G$. We may thus suppose that $\pi(x) = (1,0)$ and $\pi(y) = (0,1)$. We have an exact sequence

$$1 \longrightarrow \mathrm{IA}_2 \longrightarrow \Gamma^+(G,\pi) \xrightarrow{\rho} \Gamma(m,n) \longrightarrow 1.$$

By Proposition 1.3, the group $\Gamma(m,n)$ is free of rank

$$r := 1 + \frac{nm^2}{12} \prod_{p|m} \left(1 - \frac{1}{p^2}\right).$$

Let $\{M_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \mid 1 \leq i \leq r\}$ be a set of free generators of $\Gamma(m,n)$ and write $F_2 = \langle x, y \rangle$. Moreover, let $\varphi_i \in \mathrm{Aut}^+(F_2)$ such that $\rho(\varphi_i) = M_i$, that is

$$\varphi_i(x) \equiv a_i x + c_i y, \quad \varphi_i(y) \equiv b_i x + d_i y \mod F_2' \tag{3.1}$$

where F_2' denotes the commutator subgroup of F_2. By a classical result of J. Nielsen [8, Sec. 3.6, Cor. N4] the group IA_2 is free on α_x, α_y, the inner automorphisms given by conjugation with x and y, respectively. Now a result of P. Hall, see [6, Ch. 13, Thm. 1], yields that $\Gamma^+(G,\pi)$ admits a presentation

$$\langle \alpha_x, \alpha_y, \varphi_1, \ldots, \varphi_r \mid \varphi_i \alpha_x \varphi_i^{-1} = w_i,\ \varphi_i \alpha_y \varphi_i^{-1} = v_i \text{ for } 1 \leq i \leq r \rangle$$

where the w_i and v_i are suitable words in α_x, α_y. We have

$$\varphi_i \alpha_x \varphi_i^{-1} = \alpha_{\varphi_i(x)}, \quad \varphi_i \alpha_y \varphi_i^{-1} = \alpha_{\varphi_i(y)}$$

for $1 \leq i \leq r$. Hence, from (3.1) it follows that

$$\overline{\alpha_x} = a_i \overline{\alpha_x} + c_i \overline{\alpha_y}, \quad \overline{\alpha_y} = b_i \overline{\alpha_x} + d_i \overline{\alpha_y} \tag{3.2}$$

in the abelianization of $\Gamma^+(G,\pi)$. This yields that $\Gamma^+(G,\pi)^{\mathrm{ab}}$ is the abelian group generated by $\overline{\alpha_x}, \overline{\alpha_y}$ and $\overline{\varphi_i}, 1 \leq i \leq r$, subject to the relations (3.2). Observe that $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right) \in \Gamma(m,n)$. Hence this matrix is a product of the M_i and one consequence of the relations (3.2) is

$$\overline{\alpha_x} = \overline{\alpha_x} + n\overline{\alpha_y}.$$

We thus find that $n\overline{\alpha_y} = 0$. Similarly we find that $m\overline{\alpha_x} = 0$. Obviously we can rewrite the defining relations (3.2) as

$$(a_i - 1)\overline{\alpha_x} = c_i\overline{\alpha_y}, \quad (1 - d_i)\overline{\alpha_y} = b_i\overline{\alpha_x}. \tag{3.3}$$

By definition of $\Gamma(m, n)$ we have

$$(a_i - 1) \equiv b_i \equiv 0 \mod m, \quad (1 - d_i) \equiv c_i \equiv 0 \mod n$$

for $1 \le i \le r$ so that all relations in (3.3) are consequences of $m\overline{\alpha_x} = 0$ and $n\overline{\alpha_y} = 0$. Hence we obtain a presentation

$$\Gamma^+(G, \pi)^{\mathrm{ab}} = \langle \overline{\alpha_x}, \overline{\alpha_y}, \overline{\varphi_1}, \ldots, \overline{\varphi_r} \mid \text{abelian},\ m\overline{\alpha_x} = 0,\ n\overline{\alpha_y} = 0 \rangle.$$

This proves $\Gamma^+(G, \pi)^{\mathrm{ab}} \cong G \times \mathbb{Z}^r$. The three special cases can be obtained by an explicit computation.

3.2. **Proof of Theorem 1.2.** Let G be a finite non-perfect group. First we consider the case that $G/G' \cong \mathbb{Z}/2\mathbb{Z}$ where G' denotes the commutator subgroup of G. Then we naturally obtain an epimorphism

$$\bar{\pi} : F_2 \xrightarrow{\pi} G \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

One easily verifies that

$$\Gamma^+(G, \pi) \le \Gamma^+(\mathbb{Z}/2\mathbb{Z}, \bar{\pi}). \tag{3.4}$$

By [1, Lem. 3.1] there is some $\varphi \in \mathrm{Aut}^+(F_2)$ such that $\bar{\pi}\varphi(x) = 1$ and $\bar{\pi}\varphi(y) = 0$. Observe that $\Gamma^+(G, \pi\varphi) = \varphi^{-1}\Gamma^+(G, \pi)\varphi$. We may therefore assume that $\bar{\pi}(x) = 1$ and $\bar{\pi}(y) = 0$. We set

$$\bar{\rho} : \mathrm{Aut}^+(F_2) \xrightarrow{\rho} \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{PSL}_2(\mathbb{Z})$$

where the second epimorphism is the natural projection. Note that $\bar{\rho}$ is onto. Since $\bar{\rho}$ induces an epimorphism $\Gamma^+(G, \pi)^{\mathrm{ab}} \to \bar{\rho}(\Gamma^+(G, \pi))^{\mathrm{ab}}$, it suffices to show that $\bar{\rho}(\Gamma^+(G, \pi))$ has infinite abelianization. By (3.4) we have

$$\bar{\rho}(\Gamma^+(G, \pi)) \le \bar{\rho}(\Gamma^+(\mathbb{Z}/2\mathbb{Z}, \bar{\pi})) = \mathrm{P}\Gamma^1(2).$$

Hence $\bar{\rho}(\Gamma^+(G, \pi))$ is a finite index subgroup of $\mathrm{P}\Gamma^1(2)$. Note that $\mathrm{P}\Gamma^1(2) = \mathrm{P}\Gamma^0(2)$. By an example of Rademacher [10, Sec. 8], we have

$$\mathrm{P}\Gamma^0(2) = \left\langle \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right\rangle * \left\langle \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \right\rangle$$

where the the first factor is infinite cyclic and the second one has order 2. The Kurosh Subgroup Theorem yields that $\bar{\rho}(\Gamma^+(G, \pi))$ is the free product of

(i) a possibly trivial free group,
(ii) certain subgroups of conjugates of $\langle \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right) \rangle$,
(iii) certain conjugates of $\left\langle \left(\begin{smallmatrix} 1 & -2 \\ 1 & -1 \end{smallmatrix}\right) \right\rangle$.

We shall prove that a free factor of type (i) or (ii) actually appears. For a contradiction, let us assume that $\bar\rho(\Gamma^+(G,\pi))$ is the free product of factors of type (iii) only. Then $\bar\rho(\Gamma^+(G,\pi))$ is generated by elements of order 2 and $\bar\rho(\Gamma^+(G,\pi))^{\mathrm{ab}} \cong (\mathbb{Z}/2\mathbb{Z})^m$ for some $m \in \mathbb{N}$. Let k be the order of G. Then the automorphism $\varphi \in \mathrm{Aut}^+(F_2)$ given by

$$\varphi(x) = xy^k,\ \varphi(y) = y$$

is an element of $\Gamma^+(G,\pi)$. Hence

$$M := \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \in \bar\rho(\Gamma^+(G,\pi)).$$

Since $M \in \langle \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right) \rangle$, one easily sees that the image of M in $\mathrm{P}\Gamma^0(2)^{\mathrm{ab}}$ has infinite order. On the other hand, the image of M in $\bar\rho(\Gamma(G,\pi))^{\mathrm{ab}}$ must have finite order. Observe that the inclusion map $\bar\rho(\Gamma^+(G,\pi)) \hookrightarrow \mathrm{P}\Gamma^0(2)$ induces a homomorphism $\bar\rho(\Gamma^+(G,\pi))^{\mathrm{ab}} \to \mathrm{P}\Gamma^0(2)^{\mathrm{ab}}$ such that the following diagram commutes.

$$\begin{tikzcd}
\bar\rho(\Gamma^+(G,\pi)) \arrow[r] \arrow[d] & \mathrm{P}\Gamma^0(2) \arrow[d] \\
\bar\rho(\Gamma^+(G,\pi))^{\mathrm{ab}} \arrow[r] & \mathrm{P}\Gamma^0(2)^{\mathrm{ab}}
\end{tikzcd}$$

This implies that the image of M in $\mathrm{P}\Gamma^0(2)^{\mathrm{ab}}$ has finite order, contradiction.

The proof for $G^{\mathrm{ab}} = \mathbb{Z}/3\mathbb{Z}$ is almost the same. In the remaining cases, we find that $\bar\rho(\Gamma^+(G,\pi))$ is a finite index subgroup of a free subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. In particular, it has infinite abelianization and so must have $\Gamma^+(G,\pi)$.

References

[1] D. Appel, E. Ribnere, *On the index of congruence subgroups of* $\mathrm{Aut}(F_n)$. J. Alg. **321** (2009), 2875–2889.

[2] M. Asada, *The faithfulness of the monodromy representations associated with certain families of algebraic curves.* J. Pure Appl. Algebra **159** (2001), no. 2–3, 123–147.

[3] K.-U. Bux, M. V. Ershov, A. S. Rapinchuk, *The congruence subgroup property of* $\mathrm{Aut}(F_2)$*: A group-theoretic proof of Asada's theorem* arXiv:0909.0304v1 (2009).

[4] H. Frasch, *Die Erzeugenden der Hauptkongruenzuntergruppen für Primzahlstufen.* Math. Ann. **108** (1933), 229–252.

[5] F. Grunewald, A. Lubotzky, *Linear Representations of the Automorphism Group of a Free Group.* Geom. Funct. Anal. **18** (2009), 1564–1608.

[6] D. L. Johnson, *Presentations of Groups.* Cambridge University Press, Cambridge, 1976.

[7] G. Jones, *Congruence and non-congruence subgroups of the modular group: a survey.* Proceedings of groups — St. Andrews 1985, 223–234, London Math. Soc. Lecture Note Ser. **121**, Cambridge University Press, Cambridge, 1986.

[8] W. Magnus, A. Karrass, D. Solitar, *Combinatorial Group Theory.* John Wiley & Sons, Inc., New York, 1966.

[9] I. Pak, *What do we know about the product replacement algorithm?* Groups and Computation III (Columbus, Ohio, 1999), Ohio State Univ. Math. Res. Inst. Publ. **8**, de Gruyter, Berlin (2001), 301–347.

[10] H. Rademacher, *Über die Erzeugenden von Kongruenzuntergruppe der Modulgruppe.* Abhandlungen Hamburg **7** (1929), 134–148.

[11] T. Satoh, *The abelianization of the congruence IA-automorphism group of a free group.* Math. Proc. Camb. Phil. Soc. **142** (2007), 239–248.

[12] T. Satoh, *Corrigendum: The abelianization of the congruence IA-automorphism group of a free group.* Math. Proc. Camb. Phil. Soc. **143** (2007), 255–256.

[13] M. Siegmund, *Generalized Torelli Groups.* Dissertation (Heinrich-Heine-Universität Düsseldorf, Germany, 2007).

Department of Mathematics, Royal Holloway College, University of London, Egham, Surrey, TW20 0EX, United Kingdom.

Current address: Mathematisches Institut der Heinrich- Heine- Universität, 40225 Düsseldorf, Germany.

E-mail address: `Daniel.Appel@uni-duesseldorf.de`